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Hopf Hypersurfaces in Complex Two-Plane Grassmannians with Generalized Tanaka–Webster $\mathfrak{D}$-Parallel Shape Operator

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Abstract In this paper, we consider a new notion of generalized Tanaka–Webster $\mathfrak{D}$-parallel shape operator for a real hypersurface in a complex two-plane Grassmannian and prove a non-existence theorem of a real hypersurface.

Keywords Complex two-plane Grassmannians, real hypersurfaces, generalized Tanaka–Webster connection, parallel shape operator, $\mathfrak{D}$-parallel shape operator

MR(2010) Subject Classification 53C40, 53C15

1 Introduction

In the geometry of real hypersurfaces in Hermitian symmetric spaces, many differential geometers have studied the characterization under special condition [4, 11]. Moreover, in complex space forms or in quaternionic space forms, they have considered new notions weaker than having parallel second fundamental form, that is, $\nabla A = 0$ (see [7, 12]).

Now as an ambient space, we introduce a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ of all complex two-dimensional linear subspaces in $\mathbb{C}^{m+2}$. Another aspect of complex Grassmann manifolds $G_2(\mathbb{C}^{m+2})$ is that they are homogeneous spaces of unitary groups and represent irreducible Hermitian symmetric spaces of rank 2. Especially this Riemannian symmetric space is the unique compact Riemannian manifold being equipped with both a Kähler structure J and a quaternionic Kähler structure $\mathfrak{J}$ not containing J. The almost contact structure vector field ξ defined by $\xi = -JN$ is said to be a *Reeb* vector field, where N denotes a local unit normal vector field of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$.

The *almost contact* 3*-structure* vector fields ξ_ν for the 3-dimensional distribution $\mathfrak{D}^\perp$ of M in $G_2(\mathbb{C}^{m+2})$ are defined by $\xi_\nu = -J_\nu N$ ($\nu = 1, 2, 3$), where J_ν denotes a canonical local basis of a quaternionic Kähler structure $\mathfrak{J}$, such that $T_xM = \mathfrak{D} \oplus \mathfrak{D}^\perp$, $x \in M$. Then, naturally we could

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consider two geometric conditions for hypersurfaces M in $G_2(\mathbb{C}^{m+2})$ that the 1-dimensional distribution $[\xi] = \mathrm{Span}\{\xi\}$ and the 3-dimensional distribution $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$ are both invariant under the shape operator A of M (see [2]). Furthermore, the Reeb vector field ξ is said to be *Hopf* if it is invariant under the shape operator A. The one-dimensional foliation of M by the integral manifolds of the Reeb vector field ξ is said to be a *Hopf foliation* of M. We say that M is *Hopf hypersurface* in $G_2(\mathbb{C}^{m+2})$ if and only if the Hopf foliation of M is totally geodesic. Using the formulas in Section 2, it can be easily checked that M is Hopf if and only if the Reeb vector field ξ is Hopf.

By using two invariant conditions mentioned above and the result in Alekseevskii [1], Berndt and Suh [2] proved the following:

Theorem 1.1 *Let M be a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then both $[\xi]$ and $\mathfrak{D}^\perp$ are invariant under the shape operator of M if and only if*

(A) *M is an open part of a tube around a totally geodesic $G_2(\mathbb{C}^{m+1})$ in $G_2(\mathbb{C}^{m+2})$, or*

(B) *m is even, say $m = 2n$, and M is an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$.*

Using Theorem 1.1, Lee and Suh [10] gave a characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ in terms of the Reeb vector field ξ as follows:

Theorem 1.2 *Let M be a connected orientable Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$. Then the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$ if and only if M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$, where the distribution $\mathfrak{D}$ denotes an orthogonal complement of $\mathfrak{D}^\perp = \mathrm{Span}\{\xi_1, \xi_2, \xi_3\}$.*

Now, instead of the Levi–Civita connection, let us consider another connection, namely, the *generalized Tanaka–Webster connection* (in short, the *g-Tanaka–Webster* connection) $\hat{\nabla}^{(k)}$ for a non-zero real number k [5, 8]. This new connection $\hat{\nabla}^{(k)}$ is defined by the naturally extended one of Tanno's generalized Tanaka–Webster connection $\hat{\nabla}$ for contact metric manifolds. Actually, Tanno [14] introduced the notion of *generalized Tanaka–Webster* connection $\hat{\nabla}$ defined on contact Riemannian manifolds from the canonical connection which coincides with the Tanaka–Webster connection if the associated CR-structure is integrable.

Using such a g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$, many geometers have studied some characterizations of real hypersurfaces in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$. Recently, Jeong et al. [5] considered g-Tanaka–Webster parallel shape operator, that is, $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any tangent vector fields X, Y on M and gave a non-existence theorem for Hopf hypersurfaces in $G_2(\mathbb{C}^{m+2})$. Moreover, in [6] the authors gave a new characterization of real hypersurfaces of Type (B) in $G_2(\mathbb{C}^{m+2})$ with another parallel notion of shape operator concerned with g-Tanaka–Webster connection, that is, $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any vector fields $X \in \mathfrak{D}^\perp$ and $Y \in TM$.

Motivated by these results, in this paper we consider another new notion of g-Tanaka–Webster parallelism for the shape operator on real hypersurfaces M in $G_2(\mathbb{C}^{m+2})$. If the shape operator A of M satisfies $(\hat{\nabla}^{(k)}_X A)Y = 0$ for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$, then the shape operator is said to be *$\mathfrak{D}$-parallel in the generalized Tanaka–Webster connection*. Naturally, we know that such kind of notion is different from the g-Tanaka–Webster $\mathfrak{D}^\perp$-parallel in [6] and weaker than the g-Tanaka–Webster parallel in [5]. Then related to the notion of $\mathfrak{D}$-parallelism, we assert the following:

Theorem 1.3 *There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection.*

2 Preliminaries

Basic materials about complex two-plane Grassmannians are well known to us (see [2, 3]). This complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$ becomes a Riemannian homogeneous space, even a Riemannian symmetric space. Using Lie algebra, we normalize g such that the maximal sectional curvature of $(G_2(\mathbb{C}^{m+2}), g)$ is eight.

A canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ consists of three local almost Hermitian structures J_ν in $\mathfrak{J}$ such that $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, where the index ν is taken modulo three. Since $\mathfrak{J}$ is parallel with respect to the Riemannian connection $\tilde{\nabla}$ of $(G_2(\mathbb{C}^{m+2}), g)$, there exist for any canonical local basis $\{J_1, J_2, J_3\}$ of $\mathfrak{J}$ three local one-forms q_1, q_2, q_3 such that

$$\tilde{\nabla}_X J_\nu = q_{\nu+2}(X)J_{\nu+1} - q_{\nu+1}(X)J_{\nu+2} \tag{2.1}$$

for all vector fields X on $G_2(\mathbb{C}^{m+2})$.

Furthermore, the Riemannian curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$ is locally given by

$$\begin{aligned}\tilde{R}(X,Y)Z = {} & g(Y,Z)X - g(X,Z)Y + g(JY,Z)JX \\ & - g(JX,Z)JY - 2g(JX,Y)JZ \\ & + \sum_{\nu=1}^{3}\{g(J_\nu Y,Z)J_\nu X - g(J_\nu X,Z)J_\nu Y - 2g(J_\nu X,Y)J_\nu Z\} \\ & + \sum_{\nu=1}^{3}\{g(J_\nu JY,Z)J_\nu JX - g(J_\nu JX,Z)J_\nu JY\},\end{aligned} \tag{2.2}$$

where $\{J_1, J_2, J_3\}$ denotes a canonical local basis of $\mathfrak{J}$.

Now, let M be a real hypersurface in $G_2(\mathbb{C}^{m+2})$, that is, a hypersurface of $G_2(\mathbb{C}^{m+2})$ with real codimension one. The induced Riemannian metric on M will also be denoted by g, and ∇ denotes the Riemannian connection of (M, g). Let N be a local unit normal vector field of M and A the shape operator of M with respect to N. Let us put

$$JX = \phi X + \eta(X)N, \quad J_\nu X = \phi_\nu X + \eta_\nu(X)N \tag{2.3}$$

for any tangent vector field X of a real hypersurface M in $G_2(\mathbb{C}^{m+2})$, where N denotes a unit normal vector field of M in $G_2(\mathbb{C}^{m+2})$. From the Kähler structure J of $G_2(\mathbb{C}^{m+2})$, there exists an almost contact metric structure (ϕ, ξ, η, g) induced on M in such a way that

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta(X) = g(X, \xi) \tag{2.4}$$

for any vector field X on M. Furthermore, let $\{J_1, J_2, J_3\}$ be a canonical local basis of $\mathfrak{J}$. Then the quaternionic Kähler structure J_ν of $G_2(\mathbb{C}^{m+2})$, together with the condition $J_\nu J_{\nu+1} = J_{\nu+2} = -J_{\nu+1}J_\nu$, induces an almost contact metric 3-structure $(\phi_\nu, \xi_\nu, \eta_\nu, g)$ on M as follows:

$$\begin{aligned}& \phi_\nu^2 X = -X + \eta_\nu(X)\xi_\nu, \quad \eta_\nu(\xi_\nu) = 1, \quad \phi_\nu\xi_\nu = 0, \quad \phi_{\nu+1}\xi_\nu = -\xi_{\nu+2}, \quad \phi_\nu\xi_{\nu+1} = \xi_{\nu+2}, \\ & \phi_\nu\phi_{\nu+1}X = \phi_{\nu+2}X + \eta_{\nu+1}(X)\xi_\nu, \quad \phi_{\nu+1}\phi_\nu X = -\phi_{\nu+2}X + \eta_\nu(X)\xi_{\nu+1}\end{aligned} \tag{2.5}$$

for any vector field X tangent to M. Moreover, from the commuting property of $J_\nu J = JJ_\nu$, $\nu = 1,2,3$, the relation between these two contact metric structures (ϕ, ξ, η, g) and $(\phi_\nu, \xi_\nu, \eta_\nu, g)$, $\nu = 1,2,3$, can be given by

$$\phi\phi_\nu X = \phi_\nu \phi X + \eta_\nu(X)\xi - \eta(X)\xi_\nu, \quad \eta_\nu(\phi X) = \eta(\phi_\nu X), \quad \phi\xi_\nu = \phi_\nu\xi. \tag{2.6}$$

On the other hand, from the parallelism of Kähler structure J, that is, $\tilde{\nabla} J = 0$ and of quaternionic Kähler structure $\mathfrak{J}$ (see (2.1)), together with Gauss and Weingarten formulas it follows that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX,Y)\xi, \quad \nabla_X \xi = \phi AX, \tag{2.7}$$

$$\nabla_X \xi_\nu = q_{\nu+2}(X)\xi_{\nu+1} - q_{\nu+1}(X)\xi_{\nu+2} + \phi_\nu AX, \tag{2.8}$$

$$(\nabla_X \phi_\nu)Y = -q_{\nu+1}(X)\phi_{\nu+2}Y + q_{\nu+2}(X)\phi_{\nu+1}Y + \eta_\nu(Y)AX - g(AX,Y)\xi_\nu. \tag{2.9}$$

Using the above expression for the curvature tensor $\tilde{R}$ of $G_2(\mathbb{C}^{m+2})$, the equation of Codazzi is given by

$$\begin{aligned}(\nabla_X A)Y - (\nabla_Y A)X = {} & \eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(X)\phi_\nu Y - \eta_\nu(Y)\phi_\nu X - 2g(\phi_\nu X, Y)\xi_\nu\} \\ & + \sum_{\nu=1}^{3}\{\eta_\nu(\phi X)\phi_\nu\phi Y - \eta_\nu(\phi Y)\phi_\nu \phi X\} \\ & + \sum_{\nu=1}^{3}\{\eta(X)\eta_\nu(\phi Y) - \eta(Y)\eta_\nu(\phi X)\}\xi_\nu. \end{aligned} \tag{2.10}$$

Now, let us introduce the notion of g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ on real hypersurfaces in Kähler manifolds (see [5, 6, 8]).

As stated in the introduction, the Tanaka–Webster connection is the canonical affine connection defined on a non-degenerate pseudo-Hermitian CR-manifold (see [13, 15]). For contact metric manifolds, their associated CR-structures are pseudo-Hermitian and strongly pseudo-convex, but they are not in general integrable. In this situation, Tanno [14] defined a new connection $\hat{\nabla}$ given by

$$\hat{\nabla}_X Y = \nabla_X Y + (\nabla_X \eta)(Y)\xi - \eta(Y)\nabla_X \xi - \eta(X)\phi Y \tag{2.11}$$

for contact metric manifolds as a generalization of the original Tanaka–Webster connection. From such a point of view, we called this new connection $\hat{\nabla}$ the g-Tanaka–Webster one. From this, we know that the g-Tanaka–Webster connection $\hat{\nabla}$ coincides with the Tanaka–Webster connection if the associated CR-structure is integrable. Moreover, since a real hypersurface M of Kähler manifolds satisfies $A\phi + \phi A = 2\phi$ if and only if M is contact metric, we have another g-Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for M as an extension of the Tanno's connection $\hat{\nabla}$. Actually, by substituting (2.7) into (2.11), the generalized Tanaka–Webster connection $\hat{\nabla}^{(k)}$ for M is defined by

$$\hat{\nabla}^{(k)}_X Y = \nabla_X Y + g(\phi AX, Y)\xi - \eta(Y)\phi AX - k\eta(X)\phi Y \tag{2.12}$$

for a non-zero real number k (see [5, 6, 8]). (Note that $\hat{\nabla}^{(k)}$ is invariant under the choice of the orientation. Namely, we may take $-k$ instead of k in (2.12) for the opposite orientation $-N$.)

3 Key Lemmas

Let us assume that M is a Hopf hypersurface in a complex two-plane Grassmann manifold $G_2(\mathbb{C}^{m+2})$ with *generalized Tanaka–Webster $\mathfrak{D}$-parallel* shape operator, that is, the shape operator A satisfies

$$(\hat{\nabla}^{(k)}_X A)Y = 0 \tag{$*$}$$

for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$.

First of all, we give the fundamental equation induced from the definition of the generalized Tanaka–Webster connection (2.12) as follows:

$$\begin{aligned}(\hat{\nabla}^{(k)}_X A)Y &= \hat{\nabla}^{(k)}_X (AY) - A(\hat{\nabla}^{(k)}_X Y)\\ &= (\nabla_X A)Y + g(\phi AX, AY)\xi - \eta(AY)\phi AX - k\eta(X)\phi AY\\ &\quad - g(\phi AX, Y)A\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y\end{aligned} \tag{3.1}$$

for any tangent vector fields X and Y on M.

Since M is a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$, by the condition ($*$) the equation (3.1) is rewritten in the form

$$\begin{aligned}&(\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha\eta(Y)\phi AX - k\eta(X)\phi AY\\ &\quad - \alpha g(\phi AX, Y)\xi + \eta(Y)A\phi AX + k\eta(X)A\phi Y = 0\end{aligned} \tag{3.2}$$

for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$.

Replacing Y by ξ in (3.2), we have

$$(\nabla_X A)\xi - \alpha\phi AX + A\phi AX = 0. \tag{3.3}$$

Moreover, since $(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX$, the equation (3.3) can be written as

$$(X\alpha)\xi = 0$$

for any vector field $X \in \mathfrak{D}$.

By taking the inner product with ξ in above equation, we have $X\alpha = 0$ for any vector field $X \in \mathfrak{D}$. From this, we obtain the following result:

Lemma 3.1 *Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection. Then the principal curvature $\alpha = g(A\xi,\ \xi)$ is constant along the direction $\mathfrak{D}$, that is, $X\alpha = 0$ for any vector field $X \in \mathfrak{D}$.*

Here, it is a main goal to show that the Reeb vector field ξ belongs to either the distribution $\mathfrak{D}$ or its orthogonal complement of $\mathfrak{D}^\perp$ such that $TM = \mathfrak{D} \oplus \mathfrak{D}^\perp$ in $G_2(\mathbb{C}^{m+2})$ when the shape operator A of M is $\mathfrak{D}$-parallel in the generalized Tanaka–Webster connection.

From now on, unless otherwise stated in the present section, we may put the Reeb vector field ξ as follows:

$$\xi = \eta(X_0)X_0 + \eta(\xi_1)\xi_1 \tag{$**$}$$

for some unit vector fields $X_0 \in \mathfrak{D}$ and $\xi_1 \in \mathfrak{D}^\perp$.

On the other hand, using the notion of geodesic Reeb flow, Berndt and Suh [2, 3] proved the following:

Lemma 3.2 *If M is a connected orientable real hypersurface in $G_2(\mathbb{C}^{m+2})$ with geodesic Reeb flow, then we have the following two equations:*

$$Y\alpha = (\xi\alpha)\eta(Y) - 4\sum_{\nu=1}^{3}\eta_\nu(\xi)\eta_\nu(\phi Y), \tag{3.4}$$

and

$$\begin{aligned}&\alpha A\phi Y + \alpha\phi AY - 2A\phi AY + 2\phi Y \\ &\quad = 2\sum_{\nu=1}^{3}\{-\eta_\nu(Y)\phi\xi_\nu - \eta_\nu(\phi Y)\xi_\nu - \eta_\nu(\xi)\phi_\nu Y + 2\eta(Y)\eta_\nu(\xi)\phi\xi_\nu + 2\eta_\nu(\phi Y)\eta_\nu(\xi)\xi\}\end{aligned} \tag{3.5}$$

for any tangent vector field Y on M.

Now, using these facts, we prove the following:

Lemma 3.3 *Let M be a Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection. Then the Reeb vector field ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.*

Proof Actually, when the smooth function $\alpha = g(A\xi,\xi)$ vanishes identically, this lemma can be verified directly from (3.4). Thus, we consider only the case that the function α is non-vanishing. By using Lemma 3.1 and putting $Y \in \mathfrak{D}$ in (3.4), it becomes

$$(\xi\alpha)\eta(Y) - 4\eta_1(\xi)\eta_1(\phi Y) = 0.$$

Since $\phi\xi_1 = \eta(X_0)\phi_1 X_0$, it follows

$$\eta(X_0)(\xi\alpha)g(X_0,Y) + 4\eta(X_0)\eta_1(\xi)g(Y,\phi_1 X_0) = 0 \tag{3.6}$$

for $Y \in \mathfrak{D}$.

First, we assume that $\xi\alpha \neq 0$. Substituting Y by X_0, the equation (3.6) yields $\eta(X_0)(\xi\alpha) = 0$. By using our assumption $\xi\alpha \neq 0$, we obtain $\eta(X_0) = 0$. From this and $(**)$, we have $\xi = \eta(\xi_1)\xi_1$ and thus we assert that the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^\perp$ for this case.

Next, we suppose that $\xi\alpha = 0$. Since $X_0 \in \mathfrak{D}$, the vector field $\phi_1 X_0$ also belongs to the distribution $\mathfrak{D}$, that is, $\phi_1 X_0 \in \mathfrak{D}$. Thus substituting Y by $\phi_1 X_0$ in (3.6), we get $\eta(X_0)\eta_1(\xi) = 0$, that is, $\eta(X_0) = 0$ or $\eta_1(\xi) = 0$. It means that ξ belongs either to the distribution $\mathfrak{D}$ or the distribution $\mathfrak{D}^\perp$.

Accordingly, summing up these cases the proof of our Lemma 3.3 is completed.

4 Proof of Theorem 1.3

Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection, that is, the shape operator A satisfies the following condition:

$$(\hat{\nabla}^{(k)}_X A)Y = 0 \tag{$*$}$$

for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$. Then by Lemma 3.3, we are able to consider the following two cases that the Reeb vector field ξ either belongs to the distribution $\mathfrak{D}^\perp$ or to the distribution $\mathfrak{D}$.

First of all, we consider the case $\xi \in \mathfrak{D}^\perp$. Without loss of generality, we may put $\xi = \xi_1$.

Lemma 4.1 *If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}^{\perp}$, then there does not exist any Hopf hypersurface M in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection.*

Proof By using (3.2), we have

$$(\nabla_X A)Y + g(\phi AX, AY)\xi - \alpha g(\phi AX, Y)\xi = 0 \tag{4.1}$$

for any tangent vector fields $X, Y \in \mathfrak{D}$.

Interchanging X with Y in above equation, we get

$$(\nabla_Y A)X + g(\phi AY, AX)\xi - \alpha g(\phi AY, X)\xi = 0 \tag{4.2}$$

for any vector fields $X, Y \in \mathfrak{D}$.

By means of the equation of Codazzi (2.10) and (3.5), subtracting (4.2) from (4.1), we obtain

$$\begin{aligned} 0 &= (\nabla_X A)Y - (\nabla_Y A)X + 2g(A\phi AX, Y)\xi - \alpha g(\phi AX, Y)\xi + \alpha g(\phi AY, X)\xi \\ &= -2g(\phi X, Y)\xi - 2g(\phi_1 X, Y)\xi_1 - 2g(\phi_2 X, Y)\xi_2 - 2g(\phi_3 X, Y)\xi_3 + \alpha g(A\phi X, Y)\xi \\ &\quad + \alpha g(\phi AX, Y)\xi + 2g(\phi X, Y)\xi + 2g(\phi_1 X, Y)\xi - \alpha g(\phi AX, Y)\xi + \alpha g(\phi AY, X)\xi \end{aligned}$$

for any vector fields $X, Y \in \mathfrak{D}$.

Taking the inner product with ξ_2, the equation (4.3) reduces to

$$g(\phi_2 X, Y) = 0 \tag{4.3}$$

for any vector fields $X, Y \in \mathfrak{D}$.

Let $\{e_1,\ e_2, \ldots,\ e_{4m-4},\ e_{4m-3},\ e_{4m-2},\ e_{4m-1}\}$ be an orthonormal basis for a tangent vector space T_xM at any point $x \in M$. Without loss of generality, we may put $e_{4m-3} = \xi_1,\ e_{4m-2} = \xi_2$ and $e_{4m-1} = \xi_3$. For this orthonormal basis $\{e_i \,|\, i = 1, 2, \ldots, 4m-1\}$ the tangent vector $\phi_2 X$ where $X \in \mathfrak{D}$ is given by

$$\begin{aligned} \phi_2 X &= \sum_{\nu=1}^{4m-1} g(\phi_2 X, e_i)e_i \\ &= \sum_{\nu=1}^{4m-4} g(\phi_2 X, e_i)e_i + \sum_{\nu=1}^{3} g(\phi_2 X, \xi_\nu)\xi_\nu. \end{aligned}$$

Besides, since $g(\phi_2 X, \xi_\nu) = 0$ for any $\nu = 1, 2, 3$, we see that $\phi_2 X \in \mathfrak{D}$ for $X \in \mathfrak{D}$. Therefore, from (4.3) it implies that

$$\phi_2 X = 0$$

for $X \in \mathfrak{D}$. Applying ϕ_2 to both sides, we get $X = \eta_2(X)\xi_2$ for any tangent vector field $X \in \mathfrak{D}$. Consequently it follows that any tangent vector X belonging to the distribution $\mathfrak{D}$ becomes zero, that is, it means that $\dim M = 3$. But since the dimension of M is $4m-1$ ($m \geq 3$), it makes a contradiction. So, we can assert our Lemma 4.1.

Next we consider the case $\xi \in \mathfrak{D}$. By virtue of Theorem 1.2 due to Lee and Suh [10], we give the following:

Lemma 4.2 *Let M be a Hopf hypersurface in $G_2(\mathbb{C}^{m+2})$ with $\mathfrak{D}$-parallel shape operator in the generalized Tanaka–Webster connection. If the Reeb vector field ξ belongs to the distribution $\mathfrak{D}$, then M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{H}P^n$ in $G_2(\mathbb{C}^{m+2})$, $m = 2n$.*

From the above two Lemmas 4.1, 4.2 and the classification theorem given by Theorem 1.1 in this paper, we see that M is locally congruent to a model space of Type (B) in Theorem 1.1 under the assumption of our Theorem 1.3 given in Introduction.

Hence it remains to check if the shape operator A of real hypersurfaces of Type (B) satisfies the condition $(*)$ for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$. In order to do so, we introduce a proposition related to eigenspaces of the model space of Type (B) with respect to the shape operator. As the following proposition (see [2]) is well known, a real hypersurface M of Type (B) has five distinct constant principal curvatures as follows:

Proposition 4.3 *Let M be a connected real hypersurface in $G_2(\mathbb{C}^{m+2})$. Suppose that $A\mathfrak{D} \subset \mathfrak{D}$, $A\xi = \alpha\xi$, and ξ is tangent to $\mathfrak{D}$. Then the quaternionic dimension m of $G_2(\mathbb{C}^{m+2})$ is even, say $m = 2n$, and M has five distinct constant principal curvatures*

$$\alpha = -2\tan(2r), \quad \beta = 2\cot(2r), \quad \gamma = 0, \quad \lambda = \cot(r), \quad \mu = -\tan(r)$$

with some $r \in (0, \pi/4)$. The corresponding multiplicities are

$$m(\alpha) = 1, \quad m(\beta) = 3 = m(\gamma), \quad m(\lambda) = 4n - 4 = m(\mu)$$

and the corresponding eigenspaces are

$$\begin{aligned} &T_\alpha = \mathbb{R}\xi = \operatorname{Span}\{\xi\}, \\ &T_\beta = \mathfrak{J}J\xi = \operatorname{Span}\{\xi_\nu \mid \nu = 1,2,3\}, \\ &T_\gamma = \mathfrak{J}\xi = \operatorname{Span}\{\phi_\nu\xi \mid \nu = 1,2,3\}, \\ &T_\lambda, \quad T_\mu, \end{aligned}$$

where

$$T_\lambda \oplus T_\mu = (\mathbb{H}\mathbb{C}\xi)^\perp, \quad \mathfrak{J}T_\lambda = T_\lambda, \quad \mathfrak{J}T_\mu = T_\mu, \quad JT_\lambda = T_\mu.$$

The distribution $(\mathbb{H}\mathbb{C}\xi)^\perp$ is the orthogonal complement of $\mathbb{H}\mathbb{C}\xi$, where

$$\mathbb{H}\mathbb{C}\xi = \mathbb{R}\xi \oplus \mathbb{R}J\xi \oplus \mathfrak{J}\xi \oplus \mathfrak{J}J\xi.$$

To check this problem, we suppose that M has $\mathfrak{D}$-parallel shape operator with respect to the g-Tanaka–Webster connection. Putting $X = \xi \in \mathfrak{D}$, $Y = \xi_2 \in T_\beta$ in (3.2) and using (2.5), it becomes

$$\begin{aligned} 0 &= (\nabla_\xi A)\xi_2 + g(\phi A\xi, A\xi_2)\xi - \alpha\eta(\xi_2)\phi A\xi - k\eta(\xi)\phi A\xi_2 \\ &\quad - \alpha g(\phi A\xi, \xi_2)\xi + \eta(\xi_2)A\phi A\xi + k\eta(\xi)A\phi\xi_2 \\ &= \alpha\beta\phi_2\xi - k\beta\phi_2\xi + k\gamma\phi_2\xi \\ &= \beta(\alpha - k)\phi_2\xi, \end{aligned}$$

because $(\nabla_\xi A)\xi_2 = \beta\nabla_\xi\xi_2 - A\nabla_\xi\xi_2$ and $\gamma = 0$. Taking the inner product with $\phi_2\xi$, we have

$$\beta(\alpha - k) = 0.$$

Since $\beta \neq 0$ by virtue of Proposition 4.3, it follows that

$$\alpha = k. \tag{4.4}$$

On the other hand, putting $X = \xi \in \mathfrak{D}$ and $Y \in T_\lambda$ in (3.2), we get

$$0 = (\nabla_\xi A)Y - k\phi AY + kA\phi Y. \tag{4.5}$$

Using the equation of Codazzi (2.10), we know

$$\begin{aligned}(\nabla_\xi A)Y &= (\nabla_Y A)\xi + \phi Y \\ &= \alpha\phi AY - A\phi AY + \phi Y.\end{aligned}$$

Thus since $Y \in T_\lambda$ and $\phi Y \in T_\mu$, the equation (4.5) can be written as

$$0 = \alpha\lambda\phi Y - \lambda\mu\phi Y + \phi Y - k\lambda\phi Y + k\mu\phi Y. \tag{4.6}$$

Therefore, inserting (4.4) in (4.6), we have

$$\begin{aligned}0 &= \alpha\lambda\phi Y - \lambda\mu\phi Y + \phi Y - \alpha\lambda\phi Y + \alpha\mu\phi Y \\ &= -\lambda\mu\phi Y + \phi Y + \alpha\mu\phi Y.\end{aligned}$$

Taking the inner product with ϕY, we obtain

$$\begin{aligned}0 &= \alpha\mu - \lambda\mu + 1 \\ &= \frac{4\tan^2(r) + 2 - 2\tan^2(r)}{1 - \tan^2(r)},\end{aligned}$$

because $\alpha = -2\tan(2r)$, $\lambda = \cot(r)$ and $\mu = -\tan(r)$ with some $r \in (0, \pi/4)$, from Proposition 4.3. Thus we get $\tan^2(r) = -1$. This gives a contradiction. So this case cannot occur.

Hence summing up these assertions, we give a complete proof of our Theorem 1.3 in Introduction. □

On the other hand, in a Levi–Civita connection, if we consider a new notion of $\mathfrak{D}$-parallel shape operator, that is,

$$(\nabla_X A)Y = 0 \tag{$*'$}$$

for any vector fields $X \in \mathfrak{D}$ and $Y \in TM$, then its notion is different from the g-Tanaka–Webster $\mathfrak{D}$-parallel and much weaker than parallel shape operator. Now using such a notion in usual Levi–Civita connection, we assert the following (see [9]):

Remark 4.4 There does not exist any Hopf hypersurface in a complex two-plane Grassmannian $G_2(\mathbb{C}^{m+2})$, $m \geq 3$, with $\mathfrak{D}$-parallel shape operator.

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